

Separating a chart

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Abstract

In this paper, we shall show a condition for that a chart is C-move equivalent to the product of two charts, the union of two charts Γ^* and Γ^{**} which are contained in disks D^* and D^{**} with $D^* \cap D^{**} = \emptyset$.

1 Introduction

Charts are oriented labeled graphs in a disk which represent surface braids (see [1],[5], and see Section 2 for the precise definition of charts, see [5, Chapter 14] for the definition of surface braids). In a chart there are three kinds of vertices; white vertices, crossings and black vertices. A C-move is a local modification of charts in a disk. The closures of surface braids are embedded closed oriented surfaces in 4-space \mathbb{R}^4 (see [5, Chapter 23] for the definition of the closures of surface braids). A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. In this paper, we shall show a condition for that a chart is C-move equivalent to the product of two charts (Theorem 1).

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [3] and [13], we investigated minimal charts with exactly four white vertices. In [14], we showed that there is no minimal chart with exactly five vertices. Hasegawa proved that there exists a minimal chart with exactly six white vertices which represents the surface braid whose closure is ambient isotopic to a 2-twist spun trefoil [2]. In [12] and [6], we investigated minimal charts with exactly six white vertices. We show that there is no minimal chart with exactly seven vertices ([7], [8], [9], [10], [11]). Thus the next targets are minimal charts with eight or nine white vertices. As an application of Theorem 1, we shall show that there are 12 kinds of types for a minimal chart with eight white vertices (Corollary 2 and Corollary 3), and there are 15 kinds of types for a minimal chart with nine white vertices (Corollary 5).

Let Γ be a chart. For each label m , we denote by Γ_m the 'subgraph' of Γ consisting of all the edges of label m and their vertices.

Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

Now we define a type of a chart: Let Γ be a chart, m a label of Γ , and n_1, n_2, \dots, n_p integers. The chart Γ is said to be of *type* $(m; n_1, n_2, \dots, n_p)$, or of *type* (n_1, n_2, \dots, n_p) briefly, if it satisfies the following three conditions:

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- (i) For each $i = 1, 2, \dots, p$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.
- (ii) If $i < 0$ or $i > p$, then Γ_{m+i} does not contain any white vertices.
- (iii) Each of the two subgraphs Γ_m and Γ_{m+p} contains at least one white vertex.

Note that $n_1 \geq 1$ and $n_p \geq 1$ by Condition (iii).

Two C-move equivalent charts Γ and Γ' are said to be *same C-type* provided that the types of the two charts are same.

For a subset X of a chart, let

$w(X)$ = the number of white vertices of the chart contained in X .

A chart Γ is *zero at label k* provided that

- (i) $\Gamma_k \cap \Gamma_{k+1} = \emptyset$,
- (ii) there exists a label i with $i \leq k$ and $w(\Gamma_i) \neq 0$, and
- (iii) there exists a label j with $k < j$ and $w(\Gamma_j) \neq 0$.

Let Γ be an n -chart, and D_1, D_2 disjoint disks with $D_i \cap \Gamma \neq \emptyset$ for $i = 1, 2$, $\partial D_i \cap \Gamma = \emptyset$ for $i = 1, 2$, and $D_1 \cup D_2 \supset \Gamma$. Then we can consider $\Gamma^* = D_1 \cap \Gamma$ and $\Gamma^{**} = D_2 \cap \Gamma$ as n -charts. Then we call the chart Γ the *product of the two charts Γ^* and Γ^{**}* .

A chart Γ is *separable at label k* if there exist subcharts Γ^*, Γ^{**} such that

- (i) Γ is the product of the two charts Γ^* and Γ^{**} ,
- (ii) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$,
- (iii) $w(\Gamma_i^*) = 0$ for all label i with $k < i$, and
- (iv) $w(\Gamma_i^{**}) = 0$ for all label i with $i \leq k$.

The following is our main theorem:

Theorem 1 *A chart Γ is zero at label k if and only if there exists a chart Γ' with the same C-type of Γ such that Γ' is separable at label k .*

A chart Γ is *minimal* if it possesses the smallest number of white vertices among the charts C-move equivalent to the chart Γ (cf. [4]).

Corollary 2 *Let Γ be a minimal chart with $w(\Gamma) = 8$. If Γ is zero at some label, then Γ is C-move equivalent to one of the following charts:*

- (a) *the product of two charts of type (4).*
- (b) *the product of two charts of type (2, 2).*

(c) the product of a chart of type (4) and a chart of type (2, 2).

Corollary 3 *Let Γ be a minimal n -chart with $w(\Gamma) = 8$ such that Γ is not zero at any label. If necessary we change all the edges of label k to ones of label $n - k$ for each $k = 1, 2, \dots, n - 1$ simultaneously, then the type of Γ is (8), (6, 2), (5, 3), (4, 4), (4, 2, 2), (3, 3, 2), (3, 2, 3), (2, 4, 2) or (2, 2, 2, 2).*

Corollary 4 *If Γ is a minimal chart with $w(\Gamma) = 9$ or 11, then the chart is not zero at any label. Namely if the type of the chart is $(m; n_1, n_2, \dots, n_p)$, then for each $i = 1, 2, \dots, p$, we have $n_i \neq 0$.*

Corollary 5 *Let Γ be a minimal n -chart with $w(\Gamma) = 9$. If necessary we change all the edges of label k to ones of label $n - k$ for each $k = 1, 2, \dots, n - 1$ simultaneously, then the type of Γ is (9), (7, 2), (6, 3), (5, 4), (5, 2, 2), (4, 3, 2), (4, 2, 3), (4, 1, 4), (3, 4, 2), (3, 3, 3), (2, 5, 2), (4, 1, 2, 2), (3, 2, 2, 2), (2, 3, 2, 2) or (2, 2, 1, 2, 2).*

The paper is organized as follows. In Section 2, we define charts. In Section 3 and Section 4, we show lemmata to need a proof of Theorem 1. In Section 5, we define ω_k -minimal charts. By using ω_k -minimal charts, we show that if a chart Γ is zero at label k , then there exists a chart Γ^* obtained from Γ by C-I-R2 moves and C-I-M2 moves such that $\Gamma_k^* \supset \Gamma_k$ and all of black vertices and white vertices in $\bigcup_{i=k+1}^{\infty} \Gamma_i^*$ are contained in the same complementary domain of Γ_k . In Section 6, we give a proof of Theorem 1 and proofs of corollaries.

2 Preliminaries

Let n be a positive integer. An n -chart is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

- (i) Every vertex has degree 1, 4, or 6.
- (ii) The labels of edges are in $\{1, 2, \dots, n - 1\}$.
- (iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i + 1$ alternately for some i , where the orientation and the label of each arc are inherited from the edge containing the arc.
- (iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i - j| > 1$.

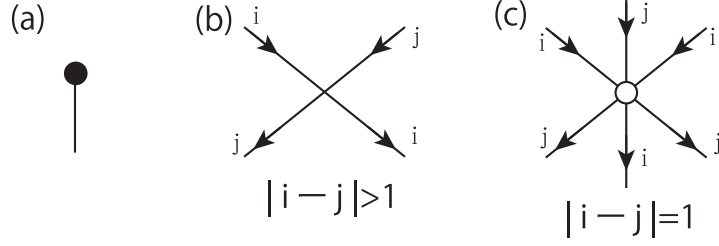


Figure 1: (a) A black vertex. (b) A crossing. (c) A white vertex.

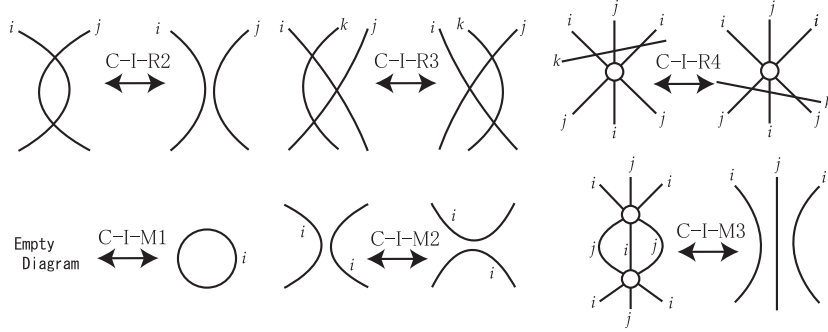


Figure 2:

We call a vertex of degree 1 a *black vertex*, a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively (see Fig. 1).

Now *C-moves* are local modifications of charts in a disk as shown in Fig. 2 (see [1], [5], [15] for the precise definition). These C-moves as shown in Fig. 2 are examples of C-moves. We often use C-I-M2 moves, and C-I-R2 moves in this paper.

Let Γ be a chart, and m a label of Γ . The 'subgraph' Γ_m of Γ consists of all the edges of label m and their vertices. An edge of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ . On the other hand, we assume that

an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m .

Thus any vertex of Γ_m is a black vertex or a white vertex. Hence any crossing of Γ is not considered as a vertex of Γ_m .

In this paper for a set X in a space we denote the interior of X , the boundary of X and the closure of X by $\text{Int}X$, ∂X and $Cl(X)$ respectively.

3 Separating systems

In this paper we assume that *every chart is in the plane*.

A disk D is *in general position* with respect to a chart Γ provided that

- (i) ∂D does not contain any white vertices, black vertices, nor crossings of Γ , and
- (ii) if an edge of Γ intersects ∂D , then the edge intersects ∂D transversely.

Let D be a disk. The simple arc ℓ is called a *proper arc* of D provided that $\ell \cap \partial D = \partial \ell$. Let L be a simple arc on ∂D . A proper arc ℓ of D is called a (D, L) -arc provided that $\partial \ell \subset L$.

Let Γ be a chart, and k a label of Γ . A simple arc in an edge of Γ_k is called an *arc of label k* .

Let D be a disk in general position with respect to a chart Γ , and L a simple arc on ∂D . A (D, L) -arc ℓ of label k is said to be *reducible* if for the subarc L' of L with $\partial L' = \partial \ell$ we have $\text{Int} L' \cap (\Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}) = \emptyset$.

For a subset X of a chart Γ , let

$\mathbb{BW}(X)$ = the set of all the black vertices and white vertices of Γ in X .

Let k be a positive integer. Let D^- and D^+ be disks in general position with respect to a chart Γ such that $J = D^- \cap D^+$ is an arc. The triplet (D^-, D^+, J) is called a *separating system at label k* for the chart Γ provided that

- (i) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i) = \emptyset$,
- (ii) $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma_i) = \emptyset$,
- (iii) $J \cap \Gamma_k = \emptyset$, and
- (iv) $Cl(\partial D^+ - J) \cap \Gamma = \emptyset$.

Lemma 6 *Let (D^-, D^+, J) be a separating system at label k for a chart Γ . Then the following hold:*

- (a) *If there exists a (D^+, J) -arc of label less than k , then there exists a reducible (D^+, J) -arc of label less than k .*
- (b) *If there exists a (D^-, J) -arc of label greater than k , then there exists a reducible (D^-, J) -arc of label greater than k .*

Proof. We show Statement (a). Suppose that there exists a (D^+, J) -arc of label less than k . Let \mathbb{S} be the set of all (D^+, J) -arcs of label less than k . For each $\ell \in \mathbb{S}$ let P_ℓ be the subarc of J with $\partial P_\ell = \partial \ell$. Let

$$m(\ell) = \text{the number of points in } \text{Int}(P_\ell) \cap (\bigcup_{i=1}^k \Gamma_i).$$

Let ℓ_0 be an element in \mathbb{S} with

$$m(\ell_0) = \min \{ m(\ell) \mid \ell \in \mathbb{S} \}.$$

Let j be the label of ℓ_0 . We have $j \leq k - 1$. Let

$s = \text{the number of points in } \text{Int}(P_{\ell_0}) \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1}).$

We show that $s = 0$ by contradiction. Suppose that $s > 0$. Let D_0 be the disk in D^+ bounded by $\ell_0 \cup P_{\ell_0}$. Let ℓ_1 be a connected component of $D_0 \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1})$ different from ℓ_0 . Since $j + 1 \leq (k - 1) + 1 = k$ and since there do not exist any black vertices nor white vertices of $\bigcup_{i=1}^k \Gamma_i$ in D^+ by Condition (ii) of a separating sysytem, the arc ℓ_1 is a proper arc of D_0 . Since $\ell_0 \cap \ell_1 = \emptyset$, the arc ℓ_1 is a (D_0, P_{ℓ_0}) -arc. Hence ℓ_1 is a (D^+, J) -arc. Let p be the label of ℓ_1 . Now $J \cap \Gamma_k = \emptyset$ implies $p \neq k$. Hence we have $p < k$. Thus $\ell_1 \in \mathbb{S}$. Since $P_{\ell_1} \cap \ell_0 = \emptyset$, we have

$$m(\ell_1) \leq m(\ell_0) - 2.$$

This contradicts that $m(\ell_0)$ is minimal. Hence we have $s = 0$. Therefore

$$\text{Int}(P_{\ell_0}) \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1}) = \emptyset.$$

This means that ℓ_0 is a desired reducible (D^+, J) -arc. This proves Statement (a).

Similarly we can show Statement (b). \square

Let Γ be a chart, and e_1 and e_2 edges of label m (possibly $e_1 = e_2$). Let α be an arc such that

- (i) $\partial\alpha$ consists of a point in e_1 and a point in e_2 , and
- (ii) $\text{Int}(\alpha)$ transversely intersects edges of Γ (see Fig. 3(a)).

Let D be a regular neighborhood of the arc α . Let $\gamma_1 = e_1 \cap D$ and $\gamma_2 = e_2 \cap D$. Then γ_1 and γ_2 are proper arcs of D and they split the disk D into three disks. Let E be the one of the three disks with $E \supset \alpha$ (see Fig. 3(b)). A chart Γ' is obtained from Γ by a surgery along α provided that

- (iii) $\Gamma'_m = (\Gamma_m - (\gamma_1 \cup \gamma_2)) \cup Cl(\partial E - (\gamma_1 \cup \gamma_2))$, and
- (iv) $\Gamma'_i = \Gamma_i$ ($i \neq m$) (see Fig. 3(c)).

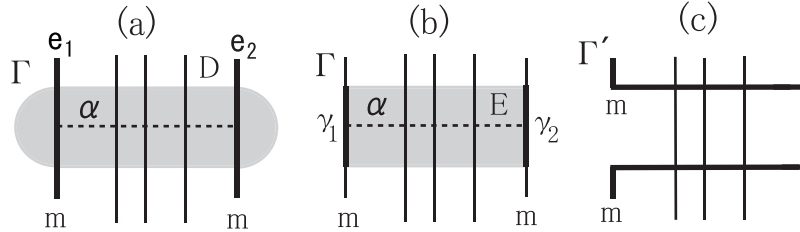


Figure 3:

Let k be a positive integer. Let Γ and Γ^* be charts. We write $\Gamma^* \stackrel{k}{\sim} \Gamma$ provided that

- (i) the chart Γ^* is obtained from Γ by applying C-I-M2 moves, C-I-R2 moves and ambient isotopies of the plane, and
- (ii) Γ_k is a subset of Γ_k^* .

For a positive integer k and a chart Γ , let

$$Fix(\Gamma_k; \Gamma) = \{ \Gamma^* \mid \Gamma^* \stackrel{k}{\sim} \Gamma \}.$$

Remark. The relation $\Gamma^* \stackrel{k}{\sim} \Gamma$ implies that Γ^* and Γ are same C-type.

Lemma 7 *Let (D^-, D^+, J) be a separating system at label k for a chart Γ . Then there exists a chart $\Gamma' \in Fix(\Gamma_k; \Gamma)$ such that*

- (a) *the chart Γ' is obtained from Γ by applying surgeries along subarcs of J , and*
- (b) *the chart Γ' does not possess any (D^-, J) -arcs of label greater than k nor (D^+, J) -arcs of label less than k .*

Proof. Let \mathbb{S} be the set of all charts obtained from Γ by applying surgeries along subarcs of J . Since $J \cap \Gamma_k = \emptyset$ by Condition (ii) of a separating system, we have that $\Gamma_k^* = \Gamma_k$ for each chart $\Gamma^* \in \mathbb{S}$. Thus for each chart $\Gamma^* \in \mathbb{S}$ we have that $\Gamma^* \in Fix(\Gamma_k; \Gamma)$ and that (D^-, D^+, J) is a separating system at label k for Γ^* . For each chart $\Gamma^* \in \mathbb{S}$, let

$$\begin{aligned} n(\Gamma^*) &= \text{the number of } (D^+, J)\text{-arcs of label less than } k \\ &\quad + \text{the number of } (D^-, J)\text{-arcs of label greater than } k. \end{aligned}$$

Let Γ' be a chart in \mathbb{S} such that

$$n(\Gamma') = \min\{ n(\Gamma^*) \mid \Gamma^* \in \mathbb{S} \}.$$

We show that $n(\Gamma') = 0$ by contradiction. Suppose that $n(\Gamma') > 0$. Then by Lemma 6 the chart Γ' possesses a reducible (D^-, J) -arc of label greater than k or a reducible (D^+, J) -arc of label less than k , say ℓ . Let P_ℓ be the subarc of J with $\partial P_\ell = \partial \ell$. Let Γ'' be a chart obtained from Γ' by applying a surgery along the subarc P_ℓ . Then we have $\Gamma'' \in \mathbb{S}$ and

$$n(\Gamma'') \leq n(\Gamma') - 1.$$

This contradicts that $n(\Gamma')$ is minimal. Therefore $n(\Gamma') = 0$. The chart Γ' is a desired chart. \square

4 Movable disks

Let D be a disk in general position with respect to a chart Γ . The disk D is called a *movable disk* at label k with respect to the chart Γ provided that

- (i) $D \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma_i) = \emptyset$, and
- (ii) $\partial D \cap (\bigcup_{i=1}^{k+1} \Gamma_i) = \emptyset$.

Let Γ be a chart, and m a label of Γ . A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). A *ring* is a closed edge of Γ_m containing a crossing but not containing any white vertices.

Lemma 8 *Let (D^-, D^+, J) be a separating system at label k for a chart Γ . If $\partial(D^- \cup D^+) \cap \Gamma_{k+1} = \emptyset$, then there exists a chart $\Gamma' \in \text{Fix}(\Gamma_k; \Gamma)$ such that*

- (a) *the chart Γ' is obtained from Γ by applying surgeries along subarcs of J , and*
- (b) *D^+ is a movable disk at label k with respect to Γ' .*

Proof. By Lemma 7 there exists a chart $\Gamma' \in \text{Fix}(\Gamma_k; \Gamma)$ obtained from Γ by applying surgeries along subarcs of J such that

- (1) Γ' does not possess any (D^-, J) -arcs of label greater than k nor (D^+, J) -arcs of label less than k .

Thus by Condition (ii), (iii) and (iv) of a separating system,

- (2) $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma'_i) = D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma_i) = \emptyset$,
- (3) $J \cap \Gamma'_k = J \cap \Gamma_k = \emptyset$,
- (4) $Cl(\partial D^+ - J) \cap \Gamma' = Cl(\partial D^+ - J) \cap \Gamma = \emptyset$.

Hence any connected component of $D^+ \cap (\bigcup_{i=1}^k \Gamma'_i)$ is a hoop, a ring or a (D^+, J) -arc of label less than k . Since Γ' does not possess any (D^+, J) -arcs of label less than k by (1), we have

- (5) $J \cap (\bigcup_{i=1}^k \Gamma'_i) = \emptyset$.

Since the chart Γ' is obtained from Γ by applying surgeries along subarcs of J , we have

- (6) $\partial(D^- \cup D^+) \cap \Gamma'_{k+1} = \partial(D^- \cup D^+) \cap \Gamma_{k+1} = \emptyset$, and
- (7) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) = D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i) = \emptyset$ by Condition (i) of a separating system.

Hence any connected component of $D^- \cap \Gamma'_{k+1}$ is a hoop, a ring or a (D^-, J) -arc. Since Γ' does not possess any (D^-, J) -arcs of label greater than k by (1), we have $J \cap \Gamma'_{k+1} = \emptyset$. Thus (4) and (5) imply

$$\partial D^+ \cap (\bigcup_{i=1}^{k+1} \Gamma'_i) = \emptyset.$$

Therefore D^+ is a movable disk at label k with respect to Γ' . \square

Let E be a disk in general position with respect to a chart Γ . The disk E is called a *c-disk at label k* with respect to Γ provided that in $\text{Int} E$ there exist mutually disjoint movable disks D_1, D_2, \dots, D_s at label k with respect to the chart Γ and a connected component W of $E - (\Gamma_k - \bigcup_{i=1}^s D_i)$ (see Fig. 4) such that

- (i) $W \supset \partial E$,
- (ii) $W \supset \bigcup_{i=1}^s D_i$, and
- (iii) $W \supset E \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^{\infty} \Gamma_i)$.

We call W the *principal domain* of the c-disk E . We also call the movable disks D_1, D_2, \dots, D_s *associated movable disks* of the c-disk E .

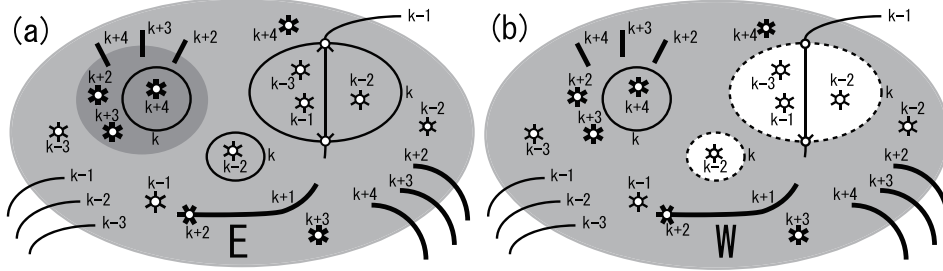


Figure 4: (a) The dark disk is a movable disk in a c-disk. (b) The gray 'disk with two holes' is the principal domain of the c-disk.

Lemma 9 *Let E be a c-disk at label k with respect to a chart Γ . Let $p \in \partial E - \Gamma$. Then there exists a separating system (D^-, D^+, J) at label k for the chart Γ with $E = D^- \cup D^+$ and $p \in \partial D^+ - J$.*

Proof. Let W be the principal domain of the c-disk E and D_1, D_2, \dots, D_s associated movable disks of E . Suppose that

$$(W - \bigcup_{i=1}^s D_i) \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^{\infty} \Gamma_i) = \{w_{s+1}, w_{s+2}, \dots, w_t\}.$$

For each $i = s+1, s+2, \dots, t$, let D_i be a regular neighbourhood of w_i in W . Then by Condition (iii) of a c-disk we have

$$(1) \quad E \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^{\infty} \Gamma_i) \subset \bigcup_{i=1}^t D_i.$$

Since W is a connected component of $E - (\Gamma_k - \bigcup_{i=1}^s D_i)$ by the condition of a c-disk, we have

$$(W - \bigcup_{i=1}^s D_i) \cap \Gamma_k = \emptyset.$$

Hence there exist mutually disjoint simple arcs $\ell_1, \ell_2, \dots, \ell_t$ in W (see Fig. 5(a)) such that

- (2) for each $i = 1, 2, \dots, t$, $\ell_i \cap \Gamma_k = \emptyset$, and $\partial \ell_i \cap \Gamma = \emptyset$,
- (3) for each $i = 1, 2, \dots, t$, the arc ℓ_i does not contain any white vertices, black vertices nor crossings of Γ , and the arc ℓ_i intersects edges of Γ transversely,
- (4) $(\bigcup_{i=1}^t \text{Int}(\ell_i)) \cap (\bigcup_{i=1}^t D_i) = \emptyset$,
- (5) for each $i = 1, 2, \dots, t-1$, the arc ℓ_i connects a point on ∂D_i and a point on ∂D_{i+1} , and
- (6) the arc ℓ_t connects a point on ∂D_t and the point p .

Let D^+ be a regular neighbourhood of $\bigcup_{i=1}^t (D_i \cup \ell_i)$ in E and $D^- = Cl(E - D^+)$. Then D^- and D^+ are disks with $E = D^- \cup D^+$. Since $E \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma_i) \subset \bigcup_{i=1}^t D_i \subset D^+$ by (1), we have

$$D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma_i) = \emptyset \text{ (see Fig. 5(b)).}$$

Since $w_{s+1}, w_{s+2}, \dots, w_t \in W - \bigcup_{i=1}^s D_i \subset E - \Gamma_k$, we have $w_{s+1}, w_{s+2}, \dots, w_t \notin \Gamma_k$. Thus

- (7) none of $D_{s+1}, D_{s+2}, \dots, D_t$ intersect $\bigcup_{i=1}^k \Gamma_i$.

Since none of the movable disks at label k nor $D_{s+1}, D_{s+2}, \dots, D_t$ intersect $\mathbb{BW}(\bigcup_{i=1}^k \Gamma_i)$, we have

$$D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma_i) = \emptyset.$$

Let $L = D^+ \cap \partial E$. Then L is a regular neighbourhood of p in ∂E . Since $p \in \partial E - \Gamma$, we have $L \cap \Gamma = \emptyset$. Let $J = D^- \cap D^+$. Then $L = Cl(\partial D^+ - J)$. Thus $Cl(\partial D^+ - J) \cap \Gamma = L \cap \Gamma = \emptyset$. Since for each $i = 1, 2, \dots, t$, $\partial D_i \cap \Gamma_k = \emptyset$ by (2), (7) and Condition (ii) of a movable disk, we have $J \cap \Gamma_k = \emptyset$. Therefore the triplet (D^-, D^+, J) is a desired separating system at label k for Γ . \square

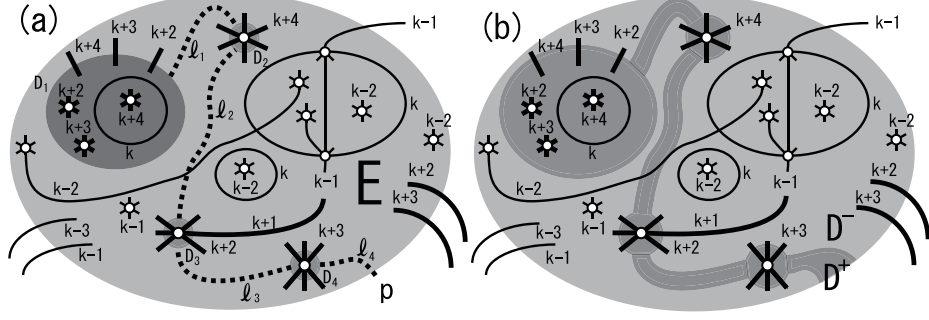


Figure 5: (a) The dark disk is a movable disk.

5 ω_k -minimal charts

Let Γ be a chart. Let Γ^* be a chart in $Fix(\Gamma_k; \Gamma)$. We denote $\Gamma^* \overset{k}{\approx} \Gamma$ provided that

for any complementary domain U of Γ_k , the domain U contains mutually disjoint movable disks D_1, D_2, \dots, D_s at label k with respect to Γ^* with $U \cap (\Gamma_k^* - \Gamma_k) \subset \bigcup_{i=1}^s D_i$.

The movable disks D_1, D_2, \dots, D_s are called *basic movable disks at label k* with respect to U and Γ^* . Let

$$\Omega(\Gamma_k; \Gamma) = \{ \Gamma^* \mid \Gamma^* \overset{k}{\approx} \Gamma \}.$$

Since $\Gamma \in \Omega(\Gamma_k; \Gamma)$, we have $\Omega(\Gamma_k; \Gamma) \neq \emptyset$.

Let Γ be a chart, k a label of Γ , and $\Gamma' \in \Omega(\Gamma_k; \Gamma)$. Let U be a complementary domain of Γ_k , and D a movable disk at label k in U with respect to Γ' . Let $F = Cl(U)$. Let $p \in \partial F - \bigcup_{i \neq k} \Gamma'_i$ and $q \in \partial D$. Suppose that *there exists a simple arc α in F connecting the two points p and q with $\alpha \cap \Gamma' = p$ and $\alpha \cap D = q$* (see Fig. 6(a)). Then we can shift the movable disk D at label k to another complementary domain of Γ_k as follows (see Fig. 6): Let N_1 be a regular neighbourhood of $D \cup \alpha$ in F . Let N_2 be a regular neighbourhood of N_1 in F and N_3 a regular neighbourhood of N_2 in F . For each $i = 1, 2, 3$ let $\beta_i = N_i \cap \partial F$ and $\gamma_i = Cl(\partial N_i - \beta_i)$ (see Fig. 6(b)). Let Γ'' be a chart with

$$\Gamma''_j = \begin{cases} (\Gamma'_k - \beta_3) \cup (\gamma_3 \cup \partial N_1) & \text{if } j = k, \\ \Gamma'_j & \text{otherwise.} \end{cases}$$

Then Γ' is C-move equivalent to Γ'' by modifying ∂F by C-I-R2 moves along γ_2 and a C-I-M2 move (see Fig. 6(c)). Let N be a regular neighbourhood of N_1 . Then N is a movable disk at label k with respect to Γ'' . In a regular neighbourhood of N_3 we can modify the arc γ_3 to the arc β_3 by an ambient isotopy keeping $\partial \gamma_3$ fixed. Let Γ^* be the resulting chart and D^* the disk modified from the movable disk N (see Fig. 6(d)). Then Γ^* is in $Fix(\Gamma_k; \Gamma)$ and D^* is a movable disk at label k with respect to Γ^* . Now Γ_k^* is the union

of Γ'_k and a ring in the movable disk D^* . Thus $\Gamma^* \stackrel{k}{\approx} \Gamma$. We say that Γ^* is obtained from Γ' by shifting the movable disk D to the outside of $Cl(U)$ along the arc α and that D^* is a movable disk induced from the movable disk D circled by a ring of label k .

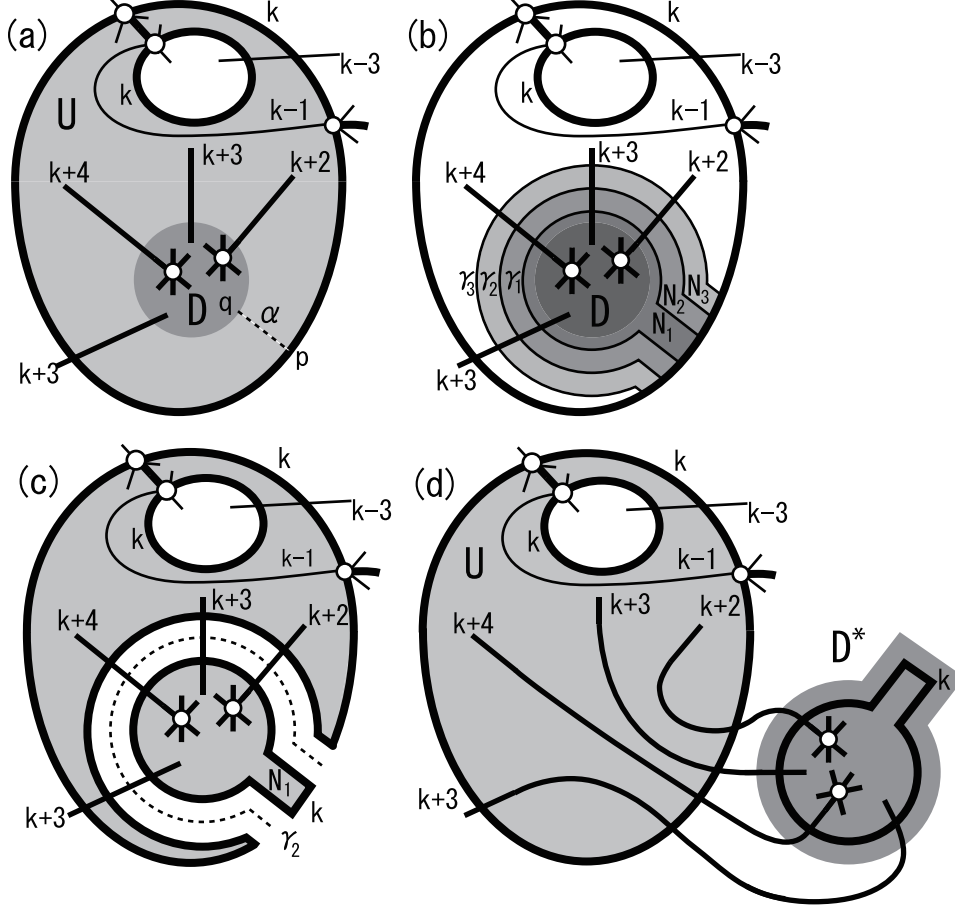


Figure 6: Shifting a movable disk.

Let Γ be a chart and k a label of Γ . Let T be a maximal tree of the dual graph of Γ_k . Namely

- (i) each vertex v of the tree T corresponds to a complementary domain U_v of Γ_k , and
- (ii) each edge e of the tree T with $\partial e = \{v_1, v_2\}$ corresponds to an edge e_Γ of Γ_k with $e_\Gamma \subset Cl(U_{v_1}) \cap Cl(U_{v_2})$.

The tree T is called a *dual tree with respect to Γ_k* .

Let Γ be a chart and k a label of Γ . Let T be a dual tree with respect to Γ_k . Let v_0 be the vertex of T which corresponds to the unbounded complementary domain of Γ_k . Let $V(T)$ be the set of all vertices of T . For each vertex $v \in V(T)$, let $T(v, v_0)$ be the path in T connecting v and v_0 . Let

$length(v; T)$ = the number of edges in $T(v, v_0)$.

We have $length(v_0; T) = 0$. For each chart $\Gamma^* \in \Omega(\Gamma_k; \Gamma)$ and $v \in V(T)$, let

$$weight_k(v; \Gamma^*) = \begin{cases} 0 & \text{if } U_v \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^{\infty} \Gamma_i^*) = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$\omega_k(\Gamma^*, T) = \sum_{v \in V(T)} weight_k(v; \Gamma^*) \times length(v; T).$$

A chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ is ω_k -minimal if there exists a dual tree T with respect to Γ_k such that $\omega_k(\Gamma', T) = \min\{ \omega_k(\Gamma^*, T) \mid \Gamma^* \in \Omega(\Gamma_k; \Gamma) \}$.

Lemma 10 *Let Γ be a chart which is zero at label k . If a chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ is ω_k -minimal, then there exists a dual tree T with respect to Γ_k with $\omega_k(\Gamma', T) = 0$.*

Proof. Suppose that for a dual tree T with respect to Γ_k , we have

$$\omega_k(\Gamma', T) = \min\{ \omega_k(\Gamma^*, T) \mid \Gamma^* \in \Omega(\Gamma_k; \Gamma) \}.$$

We shall show $\omega_k(\Gamma', T) = 0$ by contradiction. Suppose that $\omega_k(\Gamma', T) > 0$. Let $V(T)$ be the set of all vertices of T and

$$\mathbb{P} = \{ v \in V(T) \mid weight_k(v; \Gamma') \times length(v; T) > 0 \}.$$

Then $\omega_k(\Gamma', T) > 0$ implies $\mathbb{P} \neq \emptyset$. Let v_1 be a vertex in \mathbb{P} such that

$$(1) \ length(v_1; T) = \max\{ length(v; T) \mid v \in \mathbb{P} \}.$$

Then $v_1 \in \mathbb{P}$ implies

$$(2) \ 0 < weight_k(v_1; \Gamma') \times length(v_1; T) = length(v_1; T).$$

Let V_1 be a complementary domain of Γ_k corresponding to v_1 . Since $\Gamma' \in \Omega(\Gamma_k; \Gamma)$, there exist basic movable disks D_1, D_2, \dots, D_s at label k with respect to V_1 and Γ' (see Fig. 7(a)) such that

$$(3) \ V_1 \cap (\Gamma'_k - \Gamma_k) \subset \bigcup_{i=1}^s D_i.$$

Let δ be a connected component of ∂V_1 such that a bounded complementary domain of δ contains V_1 (see Fig. 7(b)). Let $N(\delta)$ be a regular neighbourhood of δ and $W = Cl(V_1 - N(\delta))$ (see Fig. 7(c)). Let ℓ be a connected component of ∂W such that W is contained in the disk E bounded by ℓ (see Fig. 7(c) and (d)). Then we have

$$W \supset \partial E.$$

Claim. The disk E is a c-disk.

Proof of Claim. Since $\Gamma' \stackrel{k}{\approx} \Gamma$, the chart Γ' is zero at label k . Hence $\delta \subset \Gamma_k \subset \Gamma'_k$ implies that $\delta \cap \Gamma'_{k+1} = \emptyset$. Thus

$$(4) \ \partial E \cap (\Gamma'_k \cup \Gamma'_{k+1}) = \emptyset.$$

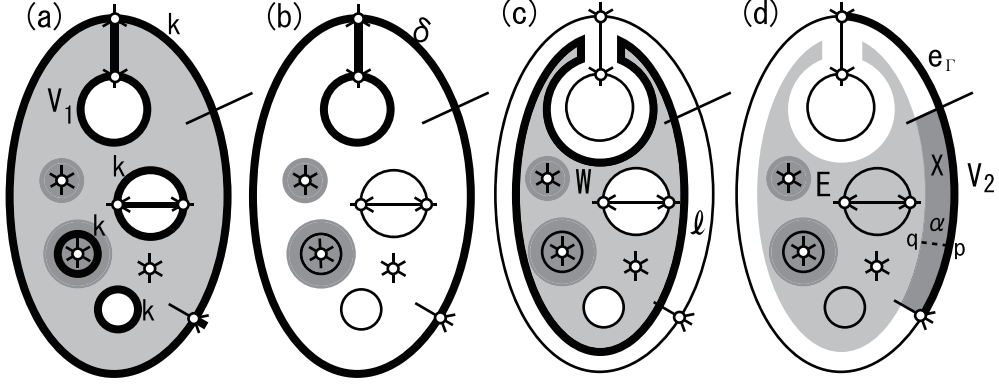


Figure 7: Dark disks are movable disks at label k . (a) The thickened curves are of label k . (b) The thickened curve is the set δ . (c) The thickened curve is the simple closed curve ℓ . (d) The thickened curve is the edge e_Γ . The set X is a disk with $\text{Int}X \cap \text{Int}E = \emptyset$.

Now $\delta \cap (\bigcup_{i=1}^s D_i) = \emptyset$ implies $N(\delta) \cap (\bigcup_{i=1}^s D_i) = \emptyset$. Thus $V_1 \supset \bigcup_{i=1}^s D_i$ implies

$$W = Cl(V_1 - N(\delta)) \supset \bigcup_{i=1}^s D_i.$$

Let \mathbb{S} be the set of all connected components of $E - (\Gamma_k - \bigcup_{i=1}^s D_i)$ different from W . Then we have

$$(5) \quad E = W \cup (\bigcup \{ Cl(U) \mid U \in \mathbb{S} \}).$$

For each domain $U \in \mathbb{S}$, let v_U be the vertex of T corresponding to U . Since U is surrounded by V_1 , we have $v_1 \in T(v_U, v_0)$ where v_0 is the vertex of T which corresponds to the unbounded complementary domain of Γ_k . Hence

$$\text{length}(v_U; T) \geq \text{length}(v_1; T) + 1 \text{ and}$$

$$\text{weight}_k(v_U; \Gamma') = 0$$

by the property (1) of the vertex v_1 . This means

$$U \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^\infty \Gamma'_i) = \emptyset.$$

Since $\Gamma' \stackrel{k}{\approx} \Gamma$, the chart Γ' is zero at label k . Thus $\Gamma'_k \cap \Gamma'_{k+1} = \emptyset$. Hence $\partial U \subset \Gamma_k \subset \Gamma'_k$ implies

$$Cl(U) \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^\infty \Gamma'_i) = \emptyset.$$

Thus (5) implies that

$$E \cap \mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^\infty \Gamma'_i) \subset W.$$

Therefore the disk E is a c-disk at label k with respect to Γ' . Hence Claim holds.

Let $T(v_1, v_0)$ be the path in the tree T connecting v_1 and v_0 . Let e be the edge of $T(v_1, v_0)$ with $e \ni v_1$. Let v_2 be the vertex of e different from v_1 . Now we have

$$(6) \text{ length}(v_2; T) = \text{length}(v_1; T) - 1.$$

Let e_Γ be the edge of Γ_k corresponding to the edge e which connects the two vertices v_1 and v_2 . Let V_2 be the complementary domain of Γ_k corresponding to v_2 . Then we have $e_\Gamma \subset Cl(V_1) \cap Cl(V_2)$. Let p be a point in $\text{Int}(e_\Gamma) - \bigcup_{i \neq k} \Gamma'_i$ and X the closure of a connected component of $(V_1 - E) - \Gamma'$ with $X \ni p$. Let q be a point in $\partial E \cap X$ and α a proper simple arc in X with $\alpha \cap \Gamma' = p$ and $\alpha \cap E = q$ (see Fig. 7(d)).

By Lemma 9 there exists a separating system (D^-, D^+, J) at label k for Γ' with $E = D^- \cup D^+$ and $q \in D^+ \cap \partial E$.

Now (4) implies

$$\partial(D^- \cup D^+) \cap \Gamma'_{k+1} = \partial E \cap \Gamma'_{k+1} = \emptyset.$$

Thus by Lemma 8 there exists a chart $\Gamma'' \in \text{Fix}(\Gamma_k; \Gamma)$ obtained from Γ' by applying surgeries along subarcs of J such that D^+ is a movable disk at label k with respect to Γ'' .

Let Γ^* be a chart obtained from Γ'' by shifting the movable disk D^+ to the outside of $Cl(V_1)$ along the arc α and let D^* be a movable disk induced from the movable disk D^+ circled by a ring of label k . Then D^* is a new movable disk in V_2 disjoint from the old movable disks in V_2 . Hence we have $\Gamma^* \in \Omega(\Gamma_k; \Gamma)$. Now

$$\text{weight}_k(v_1; \Gamma') \times \text{length}(v_1; T) = \text{length}(v_1; T) > 0 \text{ by (2),}$$

$$\text{weight}_k(v_1; \Gamma^*) \times \text{length}(v_1; T) = 0,$$

$$\text{weight}_k(v_2; \Gamma^*) \times \text{length}(v_2; T) = \text{length}(v_2; T) = \text{length}(v_1; T) - 1 \text{ by (6),}$$

and

$$\text{weight}_k(v; \Gamma^*) \times \text{length}(v; T) = \text{weight}_k(v; \Gamma') \times \text{length}(v; T) \quad (v \neq v_1, v_2)$$

imply that

$$\omega_k(\Gamma^*, T) \leq \omega_k(\Gamma', T) - 1.$$

This contradicts the fact that Γ' is ω_k -minimal. Therefore $\omega_k(\Gamma', T) = 0$. \square

6 Proof of Main Theorem

Proof of Theorem 1. Suppose that a chart Γ is zero at label k . Then there exists a chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ such that Γ' is ω_k -minimal. Since Γ is zero at label k , the chart Γ' is zero at label k , too. Thus

$$(1) \Gamma'_k \cap \Gamma'_{k+1} = \emptyset.$$

Let E be a disk containing the chart Γ' in its inside, i.e. $\text{Int}E \supset \Gamma'$. By Lemma 10 there exists a dual tree T with respect to Γ_k with $\omega_k(\Gamma', T) = 0$. Namely $\mathbb{B}\mathbb{W}(\bigcup_{i=k+1}^\infty \Gamma'_i)$ is contained in the unbounded complementary domain U of Γ_k . Now $U \cap \Gamma_k = \emptyset$. Since $\Gamma' \in \Omega(\Gamma_k; \Gamma)$, the set $U \cap (\Gamma'_k - \Gamma_k)$ is contained in the union of basic movable disks of label k with respect to U and Γ' . Furthermore, since $E \cap U$ is connected, by the same way as the one in Lemma 9 there exists a disk D^+ in E such that

- (2) D^+ is in general position with respect to Γ' ,
- (3) $D^+ \cap \partial E$ is an arc,
- (4) $\partial D^+ \cap \Gamma'_k = \emptyset$, and
- (5) $\mathbb{BW}(\Gamma') \cap D^+ = \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i)$.

Let $D^- = Cl(E - D^+)$ and $J = D^- \cap D^+$. Then we have

- (6) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) = \emptyset$ and $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma'_i) = \emptyset$ by (1) and (5),
- (7) $Cl(\partial D^+ - J) \cap \Gamma' \subset \partial E \cap \Gamma' = \emptyset$ by $\text{Int} E \supset \Gamma'$,
- (8) $J \cap \Gamma'_k \subset \partial D^+ \cap \Gamma'_k = \emptyset$ by (4).

Hence the triplet (D^-, D^+, J) is a separating system at label k for Γ' . By Lemma 7 there exists a chart $\Gamma'' \in \text{Fix}(\Gamma_k; \Gamma)$ obtained from Γ' by applying surgeries along subarcs of J such that

- (9) $\Gamma''_k = \Gamma'_k$,
- (10) Γ'' does not possess any (D^+, J) -arcs of label less than k nor (D^-, J) -arcs of label greater than k ,
- (11) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma''_i) = \emptyset$ and $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma''_i) = \emptyset$ by (6).

Now (11) and $\partial(D^- \cup D^+) \cap \Gamma'' = \partial E \cap \Gamma'' = \emptyset$ imply that

- (12) for each label i with $i < k$ if a connected component of $\Gamma''_i \cap D^+$ intersects J , then the component is a (D^+, J) -arc,
- (13) for each label i with $k < i$ if a connected component of $\Gamma''_i \cap D^-$ intersects J , then the component is a (D^-, J) -arc.

Since $J \cap \Gamma''_k = J \cap \Gamma'_k = \emptyset$ by (8) and (9), we have $J \cap \Gamma'' = \emptyset$ by (10), (12) and (13). Thus $(\partial D^+ - J) \subset \partial E$ implies

$$\partial D^+ \cap \Gamma'' = \emptyset.$$

Let $\Gamma^* = D^- \cap \Gamma''$ and $\Gamma^{**} = D^+ \cap \Gamma''$. Then Γ'' is the product of Γ^* and Γ^{**} . Now $\Gamma'' \in \text{Fix}(\Gamma_k; \Gamma)$ implies that Γ'' is zero at label k . Thus there exist two labels i and j with $i \leq k < j$, $w(\Gamma''_i) \neq 0$ and $w(\Gamma''_j) \neq 0$. Namely $w(\Gamma^*) > 0$ and $w(\Gamma^{**}) > 0$. Now (11) implies

$$w(\Gamma^*_i) = 0 \text{ for all label } i \text{ with } k < i, \text{ and}$$

$$w(\Gamma^{**}_i) = 0 \text{ for all label } i \text{ with } i \leq k.$$

Therefore Γ'' is separable at label k . It is clear that Γ'' and Γ are same C-type.

Conversely if Γ is separable at label k , then the chart Γ is clearly zero at label k . Thus we have done. \square

Lemma 11 ([8, Lemma 6.1, Proposition 6.6 and Proposition 6.11]) *Let Γ be a minimal chart of type (n_1, n_2, \dots, n_p) . Then we have the following:*

- (a) $n_1 > 1$ and $n_p > 1$.
- (b) If $n_1 = 2$ (resp. $n_p = 2$), then $n_2 > 1$ (resp. $n_{p-1} > 1$).
- (c) If $n_1 = 3$ (resp. $n_p = 3$), then $n_2 > 1$ (resp. $n_{p-1} > 1$).

Proof of Corollary 2. Let Γ be a minimal chart with $w(\Gamma) = 8$. Suppose that Γ is zero at label k . Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k . Here $w(\Gamma') = w(\Gamma)$. Hence there exist two subcharts Γ^* , Γ^{**} such that

- (1) Γ' is the product of Γ^* and Γ^{**} ,
- (2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

- (3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma') = 8$.

If either Γ^* or Γ^{**} is not minimal, then Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ'' . This contradicts the fact that Γ is minimal. Hence the two charts Γ^* and Γ^{**} are minimal. Since there does not exist a minimal chart with at most three white vertices, we have $w(\Gamma^*) \geq 4$ and $w(\Gamma^{**}) \geq 4$. Thus by (3), we have $w(\Gamma^*) = 4$ and $w(\Gamma^{**}) = 4$. By Lemma 11(a), each of Γ^* and Γ^{**} is of type (4) or (2, 2). Therefore we complete the proof of Corollary 2. \square

Proof of Corollary 3. Let Γ be a minimal n -chart with $w(\Gamma) = 8$ of type (n_1, n_2, \dots, n_p) such that Γ is not zero at any label. Then

- (1) $n_1 + n_2 + \dots + n_p = w(\Gamma) = 8$,
- (2) $n_i \geq 1$ for each i ($1 \leq i \leq p$).

By Lemma 11(a), we have $n_1 \geq 2$ and $n_p \geq 2$. If necessary we change all the edges of label k to ones of label $n - k$ for each $k = 1, 2, \dots, n - 1$ simultaneously, we can assume that

- (3) $n_1 \geq n_p \geq 2$.

There are four cases: (i) $p = 1$ or 2, (ii) $p = 3$, (iii) $p = 4$, (iv) $p \geq 5$.

Case (i). If $p = 1$, then Γ is of type (8). If $p = 2$, then by (1) and (3) we have that the chart Γ is of type (6, 2), (5, 3) or (4, 4).

Case (ii). Suppose $p = 3$. If $n_3 = 2$ or 3, then $n_2 = n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (1) and (3) we have that the chart Γ is of type (4, 2, 2), (3, 3, 2), (2, 4, 2) or (3, 2, 3).

If $n_3 \geq 4$, then by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + n_3 \geq 4 + 1 + 4 = 9$. This contradicts the fact $w(\Gamma) = 8$.

Case (iii). Suppose $p = 4$. If $n_4 = 2$ or 3 , then $n_3 = n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (1), (2) and (3) we have that the chart Γ is of type $(2, 2, 2, 2)$, $(2, 1, 3, 2)$ or $(3, 1, 2, 2)$. However if $n_1 = 2$ or 3 , then $n_2 \geq 2$ by Lemma 11(b) and (c). Hence Γ is of type $(2, 2, 2, 2)$.

If $n_4 \geq 4$, then by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + n_3 + n_4 \geq 4 + 1 + 1 + 4 = 10$. This contradicts the fact $w(\Gamma) = 8$.

Case (iv). Suppose $p \geq 5$. There are two cases: (iv-1) $n_p = 2$ or 3 , (iv-2) $n_p \geq 4$.

Cases (iv-1). We have $n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (2), (3) we have that $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_{p-1} + n_p \geq n_1 + n_2 + 1 + 2 + 2 = n_1 + n_2 + 5$. Hence

$$w(\Gamma) \geq n_1 + n_2 + 5.$$

If $n_1 = 2$, then $n_2 \geq 2$ by Lemma 11(b). Thus $w(\Gamma) \geq 2 + 2 + 5 = 9$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 3$, then by (2) we have $w(\Gamma) \geq 3 + 1 + 5 = 9$. This contradicts the fact $w(\Gamma) = 9$.

Cases (iv-2). Since $n_p \geq 4$, by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_{p-1} + n_p \geq 4 + 1 + 1 + 1 + 4 = 11$. This contradicts the fact $w(\Gamma) = 9$. \square

A chart Γ belongs to *the first class* provided that

- (i) $w(\Gamma)$ is odd, and
- (ii) there does not exist a minimal chart Γ' such that $w(\Gamma')$ is odd and less than $w(\Gamma)$.

Corollary 12 *If a minimal chart belongs to the first class, then the chart is not zero at any label. Namely if the type of the chart is $(m; n_1, n_2, \dots, n_p)$, then for each $i = 1, 2, \dots, p$, we have $n_i \neq 0$.*

Proof. Let Γ be a minimal chart belonging to the first class. Suppose that the chart is zero at label k . Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k . Thus there exist subcharts Γ^* and Γ^{**} such that

- (1) Γ' is the product of the two charts Γ^* and Γ^{**} ,
- (2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

- (3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma')$.

Since Γ' and Γ are same C-type, we have $w(\Gamma') = w(\Gamma)$. Since Γ belongs to the first class, we have that $w(\Gamma') (= w(\Gamma))$ is odd. Thus $w(\Gamma^*)$ or $w(\Gamma^{**})$ is odd less than $w(\Gamma)$. Since Γ belongs to the first class, either Γ^* or Γ^{**} is not a minimal chart. Hence Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ'' . This contradicts the fact that Γ is minimal. Therefore Γ is not zero at any label. \square

Proof of Corollary 4. There does not exist a minimal chart Γ with $w(\Gamma) = 1, 2$, or 3 . Further there does not exist a minimal chart Γ with $w(\Gamma) = 5$ ([14]). Furthermore there does not exist a minimal chart Γ with $w(\Gamma) = 7$ ([7], [8], [9], [10], [11]). Hence any chart with nine white vertices belongs to the first class. Thus any minimal chart with nine white vertices is not zero at any label by Corollary 12.

Let Γ be a minimal chart with $w(\Gamma) = 11$. Suppose that Γ is zero at label k . Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k . Here $w(\Gamma') = w(\Gamma)$. Hence there exist two subcharts Γ^*, Γ^{**} such that

- (1) Γ' is the product of the two charts Γ^* and Γ^{**} ,
- (2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

- (3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma') = 11$.

There are five cases:

- (i) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals **1** and the other equals 10,
- (ii) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals **2** and the other equals 9,
- (iii) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals **3** and the other equals 8,
- (iv) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 4 and the other equals **7**, and
- (v) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals **5** the other equals 6.

But in any case, either Γ^* or Γ^{**} is not minimal. Hence Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ'' . This contradicts the fact that Γ is minimal. Therefore any minimal chart Γ with $w(\Gamma) = 11$ is not zero at any label. \square

Proof of Corollary 5. Let Γ be a minimal n -chart with $w(\Gamma) = 9$ of type (n_1, n_2, \dots, n_p) . Then

- (1) $n_1 + n_2 + \dots + n_p = w(\Gamma) = 9$.

By Corollary 4, we have

(2) $n_i \geq 1$ for each i ($1 \leq i \leq p$).

By Lemma 11(a), we have $n_1 \geq 2$ and $n_p \geq 2$. If necessary we change all the edges of label k to ones of label $n - k$ for each $k = 1, 2, \dots, n - 1$ simultaneously, we can assume that

(3) $n_1 \geq n_p \geq 2$, and

(4) if $n_1 = n_p$ and $p \geq 4$, then $n_2 \geq n_{p-1}$.

If $p = 1$, then Γ is of type (9). If $p = 2$, then by (1) and (3) the chart Γ is of type (7, 2), (6, 3) or (5, 4).

Suppose $p \geq 3$. There are two cases: (i) $n_p = 2$ or 3, (ii) $n_p \geq 4$.

Case (i). By Lemma 11(b) and (c), we have $n_{p-1} \geq 2$. There are four cases: (i-1) $p = 3$, (i-2) $p = 4$, (i-3) $p = 5$, (i-4) $p \geq 6$.

Case (i-1). Suppose $p = 3$. By (1), (3) and $n_2 = n_{p-1} \geq 2$, the chart Γ is of type (5, 2, 2), (4, 3, 2), (3, 4, 2), (2, 5, 2), (4, 2, 3) or (3, 3, 3).

Case (i-2). Suppose $p = 4$. If $n_1 = 2$ or 3, then $n_2 \geq 2$ by Lemma 11(b) and (c). Thus by (1), (3), (4) and $n_3 = n_{p-1} \geq 2$, the chart Γ is of type (2, 3, 2, 2) or (3, 2, 2, 2).

If $n_1 = 4$, by (1), (2), (3) and $n_3 = n_{p-1} \geq 2$, then Γ is of type (4, 1, 2, 2)

If $n_1 \geq 5$, then by (2), (3) and $n_3 = n_{p-1} \geq 2$, we have $w(\Gamma) = n_1 + n_2 + n_3 + n_4 \geq 5 + 1 + 2 + 2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

Case (i-3). Suppose $p = 5$. If $n_1 = 2$, then $n_2 \geq 2$ by Lemma 11(b). Thus by (1), (2), (3) and $n_4 = n_{p-1} \geq 2$, we have that the chart Γ is of type (2, 2, 1, 2, 2).

If $n_1 = 3$, then $n_2 \geq 2$ by Lemma 11(c). Thus by (2), (3) and $n_4 = n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + n_3 + n_4 + n_5 \geq 3 + 2 + 1 + 2 + 2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 4$, then by (2), (3) and $n_4 = n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + n_3 + n_4 + n_5 \geq 4 + 1 + 1 + 2 + 2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

Case (i-4). Suppose $p \geq 6$. If $n_1 = 2$ or 3, then $n_2 \geq 2$ by Lemma 11(b) and (c). Thus by (2), (3) and $n_{p-1} \geq 2$, we have $w(\Gamma) = n_1 + n_2 + \dots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_4 + n_{p-1} + n_p \geq 2 + 2 + 1 + 1 + 2 + 2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 4$, then by (2), (3) and $n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + \dots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_4 + n_{p-1} + n_p \geq 4 + 1 + 1 + 1 + 2 + 2 = 11$. This contradicts the fact $w(\Gamma) = 9$.

Case (ii). If $p = 3$, then by (1), (2) and (3) we have Γ is of type (4, 1, 4).

If $p \geq 4$, then by (2) and (3) we have $w(\Gamma) = n_1 + n_2 + \dots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_p \geq 4 + 1 + 1 + 4 = 10$. This contradicts the fact $w(\Gamma) = 9$.

□

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